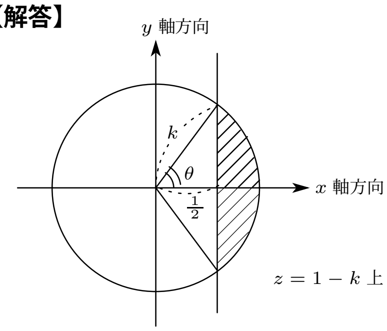
【解答】



平面 $z=1-k$ できると円錐の切り口は半径が k の円となる。$\frac{1}{2} \leqq k \leqq 1$ のとき、図の角を θ とおくと

$$k\cos\theta = \frac{1}{2} \Longleftrightarrow k = \frac{1}{2\cos\theta}$$

よって、$z=1-k$ による断面積を S とすると

$$S = k^2\theta - \frac{1}{2}\cdot k\sin\theta = \frac{\theta}{4\cos^2\theta} - \frac{\sin\theta}{4\cos\theta}$$

$dV = Sdk = S\frac{\sin\theta}{2\cos^2\theta}d\theta$ だから

$$V = \int_0^{\frac{\pi}{3}} \left(\frac{\theta}{4\cos^2\theta} - \frac{\sin\theta}{4\cos\theta}\right)\frac{\sin\theta}{2\cos^2\theta}d\theta = \frac{1}{8}\int_0^{\frac{\pi}{3}} \theta\cdot\frac{\sin\theta}{\cos^4\theta}d\theta - \frac{1}{8}\int_0^{\frac{\pi}{3}}\frac{\sin^2\theta}{\cos^3\theta}d\theta \qquad \cdots①$$

第一項の積分を J_1, 第二項の積分を J_2, $\quad I_{-3} = \int_0^{\frac{\pi}{3}}\frac{1}{\cos^3\theta}d\theta, I_{-1} = \int_0^{\frac{\pi}{3}}\frac{1}{\cos\theta}d\theta$ とすると

$$J_1 = \int_0^{\frac{\pi}{3}} \theta\left(\frac{1}{3\cos^3\theta}\right)' d\theta = \left[\theta\left(\frac{1}{3\cos^3\theta}\right)\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}}\frac{1}{3\cos^3\theta}d\theta = \frac{8\pi}{9} - \frac{1}{3}I_{-3}$$

$$J_2 = \int_0^{\frac{\pi}{3}}\frac{\sin^2\theta}{\cos^3\theta}d\theta = \int_0^{\frac{\pi}{3}}\frac{1-\cos^2\theta}{\cos^3\theta}d\theta = \int_0^{\frac{\pi}{3}}\frac{1}{\cos^3\theta}d\theta - \int_0^{\frac{\pi}{3}}\frac{1}{\cos\theta}d\theta = I_{-3} - I_{-1}$$

ゆえに

$$V = \frac{1}{8}(J_1 - J_2) = \frac{1}{8}\left\{\left(\frac{8\pi}{9} - \frac{1}{3}I_{-3}\right) - (I_{-3} - I_{-1})\right\} = \frac{\pi}{9} - \frac{I_{-3}}{6} + \frac{I_{-1}}{8} \qquad \cdots②$$

ここで

$$I_{-1} = \int_0^{\frac{\pi}{3}}\frac{1}{\cos\theta}d\theta = \int_0^{\frac{\pi}{3}}\frac{\cos\theta}{\cos^2\theta}d\theta = \int_0^{\frac{\pi}{3}}\frac{(\sin\theta)'}{1-\sin^2\theta}d\theta$$

$\sin\theta = u$ とおくと

$$I_{-1} = \int_0^{\frac{\sqrt{3}}{2}}\frac{du}{1-u^2} = \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du = \frac{1}{2}\left[\log\left|\frac{1+u}{1-u}\right|\right]_0^{\frac{\sqrt{3}}{2}} = \log(2+\sqrt{3}) \qquad \cdots③$$

また

$$I_{-3} = \int_0^{\frac{\pi}{3}}\frac{1}{\cos^3\theta}d\theta = \int_0^{\frac{\pi}{3}}\frac{(\tan\theta)'}{\cos\theta}d\theta = \left[\frac{\tan\theta}{\cos\theta}\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}}\frac{\sin\theta}{\cos^2\theta}\tan\theta d\theta$$

$$= 2\sqrt{3} - \int_0^{\frac{\pi}{3}}\frac{1-\cos^2\theta}{\cos^3\theta}d\theta = 2\sqrt{3} - (I_{-3} - I_{-1})$$

$$\therefore\ I_{-3} = \sqrt{3} + \frac{1}{2}I_{-1} = \sqrt{3} + \frac{1}{2}\log(2+\sqrt{3}) \qquad \cdots④$$

よって②,③,④ より

$$V = \frac{\pi}{9} - \frac{\sqrt{3} + \frac{1}{2}\log(2+\sqrt{3})}{6} + \frac{\log(2+\sqrt{3})}{8} = \frac{\pi}{9} - \frac{\sqrt{3}}{6} + \frac{1}{24}\log(2+\sqrt{3}) \qquad \cdots(答)$$