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(1)

$$f(z_0) = z_0 \iff z_0^2 - z_0 + 1 = z_0 \iff (z_0 - 1)^2 = 0 \iff z_0 = 1 \qquad \cdots ans.$$

(2)

$$f(z) - f(z_0) = (z^2 - z + 1) - (z_0^2 - z_0 + 1) = (z + z_0 - 1)(z - z_0)$$

Therfore if we set $\arg(z + z_0 - 1) = \theta$, $|z + z_0 - 1| = r$,then,

$$\overrightarrow{AP} \xrightarrow[\theta \text{ rotate}]{\text{rotate by the factor of } r} \overrightarrow{AQ} \qquad \cdots ①$$

Because $AP = 2$,then considering ①

$$\triangle APQ = \frac{1}{2} \cdot AP \cdot AQ \cdot \sin \angle PAQ = \frac{1}{2} \cdot 2 \cdot 2r \cdot |\sin \theta|$$

However, $r \sin \theta$ equals the imaginary value of $(z + z_0 - 1)$, then

$$\triangle APQ = 2 \times |Im(z + z_0 - 1)| = 2 \times |Im(z)| \qquad \cdots ②$$

z moves in a circumference of a circle of radius 2 with a center of $A(1)$, then $|Im(z)|$ has muximum value 2 at

$$z = 1 \pm 2i$$

Therfore, absolute maximum of the area of $\triangle APQ$ is

$$4 \text{ (if } z = 1 \pm 2i) \qquad \cdots ans.$$