$\boxed{12}$

$$\left\{\frac{\sin\theta}{(2+\cos\theta)^n}\right\}' = \frac{\cos\theta(2+\cos\theta)^n - \sin\theta\times n(2+\cos\theta)^{n-1}(-\sin\theta)}{(2+\cos\theta)^{2n}}$$
$$= \frac{\cos\theta(2+\cos\theta)+n\sin^2\theta}{(2+\cos\theta)^{n+1}}$$
$$= \frac{2\cos\theta + n - (n-1)\cos^2\theta}{(2+\cos\theta)^{n+1}} \qquad \cdots(*)$$

This expression is true when $n = 0$,so if $n = 1, 2, 3\cdots$, then

$$\left\{\frac{\sin\theta}{(2+\cos\theta)^{n-1}}\right\}' = \frac{2\cos\theta + (n-1) - (n-2)\cos^2\theta}{(2+\cos\theta)^n}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{2\cos\theta + (n-1) - (n-2)\cos^2\theta}{(2+\cos\theta)^n}d\theta = \left[\frac{\sin\theta}{(2+\cos\theta)^{n-1}}\right]_0^{\frac{\pi}{2}} = \frac{1}{2^{n-1}} \qquad \cdots ①$$

In addition,

$$\int_0^{\frac{\pi}{2}} \frac{2\cos\theta + (n-1) - (n-2)\cos^2\theta}{(2+\cos\theta)^n}d\theta$$
$$= (n-1)\int_0^{\frac{\pi}{2}} \frac{1}{(2+\cos\theta)^n}d\theta + 2\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(2+\cos\theta)^n}d\theta - (n-2)\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{(2+\cos\theta)^n}d\theta$$
$$= (n-1)I_n + 2J_n - (n-2)K_n \qquad \cdots ②$$

By means of ①,②.

$$(n-1)I_n + 2J_n - (n-2)K_n = \frac{1}{2^{n-1}} \qquad \cdots ③$$

Therefore,

$$2I_{n+1} + J_{n+1} = \int_0^{\frac{\pi}{2}} \frac{2+\cos\theta}{(2+\cos\theta)^{n+1}}d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{(2+\cos\theta)^n}d\theta = I_n \qquad \cdots ④$$
$$K_{n+1} - 4I_{n+1} = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta - 4}{(2+\cos\theta)^{n+1}}d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos\theta - 2}{(2+\cos\theta)^n}d\theta = J_n - 2I_n \qquad \cdots ⑤$$

By means of ③,④,⑤ .

$$3(n+1)I_{n+2} - (4n+2)I_{n+1} + nI_n = -\frac{1}{2^{n+1}} \ (n = 1, 2, 3\cdots) \qquad \cdots ⑥$$

Much the same as $\boxed{11}$.

$$I_1 = \frac{\sqrt{3}}{9}\pi \qquad \cdots ans.$$

If we define I_0 adequately, ⑥ holds true if $n = 0$. Substitute $n = 0, 1, 2$ to ⑥ in this order.

$$I_2 = \frac{2\sqrt{3}}{27}\pi - \frac{1}{6},\ I_3 = \frac{\sqrt{3}}{18}\pi - \frac{5}{24},\ I_4 = \frac{11\sqrt{3}}{243}\pi - \frac{5}{24} \qquad \cdots ans.$$

Comment

⑥ holds true if $n < 0$.