x^5 +15 x + 12 の厳密解 with *Mathematica* 13.3

2025 年2月 by mixedmoss

「FindGaloisGroup.nb」で最後から2番目に書いた「StepByStep Mathematica Program 」を 5次向けに直し，それを使って求めます．もちろん 最後に挙げた「Automated Mathematica Program」 の方を使うとすぐ求まりますすが，それでは説明になりません．いまさら「説明は不要」という方は C5_Solutions.nb/pdf を御覧ください．逆に，より詳しい説明は「FindGaloisGroup.nb/pdf」をご覧ください．

【注】余りにも長いので結果を一部しか表示していません．全ての結果が欲しい方は Mathematica を使ってノートブックを開いてください．また「F20_Solution . nb/pdf 」には，原始元による全ての解の表現が小さいフォントで載せてあります．

Step1 . 分解方程式

(出力は 120次の方程式R (x)を因数分解した式, そして分解方程式V(x)です．分解方程式はR(x)の他の因子を選んでも大丈夫です)

In[29]:=
```
ClearAll["`*"];
f[x_] = x^5 + 15 x + 12;
vars = {α, β, γ, δ, ϵ};
vs = Permutations[vars].{1, 4, 3, 2, 5}; (*R(x)に重解があるときはここを変える*)
coef = CoefficientList[f[x], x] // Reverse // Drop[#, 1] &;
{s1, s2, s3, s4, s5} = Table[SymmetricPolynomial[i, vars], {i, 1, 5}];
contents =
  {s1 + coef⟦1⟧, s2 - coef⟦2⟧, s3 + coef⟦3⟧, s4 - coef⟦4⟧, s5 + coef⟦5⟧, v - vs⟦1⟧};
GroebnerBasis[contents, {α, β, γ, δ, ϵ, v}]⟦1⟧;
Factor[%]
V[x_] = (List @@ %)⟦1⟧ /. {v → x}
```

Out[37]=

$$(2\,371\,842\,000\,000\,000 + 79\,290\,090\,000\,000\,v^2 - 10\,682\,052\,390\,000\,v^4 + 850\,670\,100\,000\,v^6 - 34\,046\,433\,000\,v^8 + 523\,980\,000\,v^{10} + 9\,587\,025\,v^{12} - 90\,000\,v^{14} - 5370\,v^{16} + v^{20})$$
$$(1\,422\,044\,100\,000\,000 + 138\,189\,240\,000\,000\,v - 294\,575\,454\,000\,000\,v^2 - 23\,777\,355\,600\,000\,v^3 + 11\,860\,604\,010\,000\,v^4 + 2\,551\,937\,400\,000\,v^5 + 579\,177\,000\,000\,v^6 - 152\,842\,140\,000\,v^7 - 15\,318\,423\,000\,v^8 - 4\,634\,010\,000\,v^9 + 275\,985\,000\,v^{10} + 104\,328\,000\,v^{11} + 16\,220\,025\,v^{12} + 486\,000\,v^{13} + 45\,000\,v^{14} - 43\,200\,v^{15} - 3570\,v^{16} + v^{20})$$
$$(1\,422\,044\,100\,000\,000 - 138\,189\,240\,000\,000\,v - 294\,575\,454\,000\,000\,v^2 + 23\,777\,355\,600\,000\,v^3 + 11\,860\,604\,010\,000\,v^4 - 2\,551\,937\,400\,000\,v^5 + 579\,177\,000\,000\,v^6 + 152\,842\,140\,000\,v^7 - 15\,318\,423\,000\,v^8 + 4\,634\,010\,000\,v^9 + 275\,985\,000\,v^{10} - 104\,328\,000\,v^{11} + 16\,220\,025\,v^{12} - 486\,000\,v^{13} + 45\,000\,v^{14} + 43\,200\,v^{15} - 3570\,v^{16} + v^{20})$$
$$(936\,697\,219\,200\,000 + 176\,684\,490\,000\,000\,v^2 + 12\,060\,920\,250\,000\,v^4 + 595\,066\,500\,000\,v^6 - 13\,627\,305\,000\,v^8 - 446\,400\,000\,v^{10} + 6\,152\,625\,v^{12} + 450\,000\,v^{14} - 1050\,v^{16} + v^{20})$$
$$(65\,456\,100\,000\,000 - 157\,162\,680\,000\,000\,v + 85\,979\,394\,000\,000\,v^2 + 9\,195\,022\,800\,000\,v^3 + 972\,810\,810\,000\,v^4 + 2\,375\,001\,000\,000\,v^5 + 570\,337\,200\,000\,v^6 + 31\,251\,420\,000\,v^7 + 54\,667\,197\,000\,v^8 - 417\,150\,000\,v^9 + 782\,775\,000\,v^{10} - 47\,304\,000\,v^{11} + 1\,406\,025\,v^{12} - 2\,430\,000\,v^{13} - 225\,000\,v^{14} - 14\,400\,v^{15} + 30\,v^{16} + v^{20})$$
$$(65\,456\,100\,000\,000 + 157\,162\,680\,000\,000\,v + 85\,979\,394\,000\,000\,v^2 - 9\,195\,022\,800\,000\,v^3 + 972\,810\,810\,000\,v^4 - 2\,375\,001\,000\,000\,v^5 + 570\,337\,200\,000\,v^6 - 31\,251\,420\,000\,v^7 + 54\,667\,197\,000\,v^8 + 417\,150\,000\,v^9 + 782\,775\,000\,v^{10} + 47\,304\,000\,v^{11} + 1\,406\,025\,v^{12} + 2\,430\,000\,v^{13} - 225\,000\,v^{14} + 14\,400\,v^{15} + 30\,v^{16} + v^{20})$$

Out[38]=

$$2\,371\,842\,000\,000\,000 + 79\,290\,090\,000\,000\,x^2 - 10\,682\,052\,390\,000\,x^4 + 850\,670\,100\,000\,x^6 - 34\,046\,433\,000\,x^8 + 523\,980\,000\,x^{10} + 9\,587\,025\,x^{12} - 90\,000\,x^{14} - 5370\,x^{16} + x^{20}$$

Step2 . f (x)の解を原始元vで表す

In[39]:=

```
GroebnerBasis[contents, {α, v}, {β, γ, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
α' = α /. Solve[% == 0, α]〚1〛 // Collect[#, v] &;
GroebnerBasis[contents, {β, v}, {α, γ, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
β' = β /. Solve[% == 0, β]〚1〛 // Collect[#, v] &;
GroebnerBasis[contents, {γ, v}, {α, β, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
γ' = γ /. Solve[% == 0, γ]〚1〛 // Collect[#, v] &;
GroebnerBasis[contents, {δ, v}, {α, β, γ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
δ' = δ /. Solve[% == 0, δ]〚1〛 // Collect[#, v] &;
GroebnerBasis[contents, {ϵ, v}, {α, β, γ, δ}] // PolynomialMod[Last[#], V[v]] &;
ϵ' = ϵ /. Solve[% == 0, ϵ]〚1〛 // Collect[#, v] &;
{α', ·········}
```

Out[49]=

$$
\left\{-\frac{599\,617\,885\,723\,795\,951\,453\,660\,719\,638\,739\,843}{1\,420\,320\,024\,394\,856\,731\,980\,705\,723\,752\,816\,204}-\frac{26\,187\,375\,796\,901\,565\,181\,802\,637\,171\,588\,565\,003\,v}{78\,117\,601\,341\,717\,120\,258\,938\,814\,806\,404\,891\,220}+\frac{1\,106\,404\,820\,614\,946\,076\,307\,742\,486\,406\,650\,823\,v^{2}}{40\,013\,672\,398\,551\,260\,668\,140\,952\,869\,404\,135\,200}+\frac{21\,087\,184\,232\,616\,205\,636\,146\,300\,862\,062\,737\,347\,v^{3}}{2\,200\,751\,981\,920\,319\,336\,747\,752\,407\,817\,227\,436\,000}-\frac{42\,981\,639\,224\,137\,260\,431\,777\,423\,944\,878\,945\,711\,v^{4}}{10\,998\,958\,268\,913\,770\,532\,458\,585\,124\,741\,808\,683\,776}-\frac{2\,091\,968\,726\,706\,281\,954\,908\,839\,380\,374\,141\,855\,889\,v^{5}}{3\,024\,713\,523\,951\,286\,896\,426\,110\,909\,303\,997\,388\,038\,400}+\frac{1\,836\,961\,584\,026\,471\,232\,073\,005\,104\,331\,868\,919\,761\,v^{6}}{8\,249\,218\,701\,685\,327\,899\,343\,938\,843\,556\,356\,512\,832\,000}+\frac{8\,783\,108\,814\,978\,501\,593\,653\,695\,738\,320\,478\,566\,719\,v^{7}}{453\,707\,028\,592\,693\,034\,463\,916\,636\,395\,599\,608\,205\,760\,000}-\frac{22\,264\,772\,492\,237\,136\,377\,943\,469\,680\,312\,399\,431\,v^{8}}{3\,299\,687\,480\,674\,131\,159\,737\,575\,537\,422\,542\,605\,132\,800}+\frac{129\,567\,878\,619\,851\,820\,842\,416\,854\,801\,665\,719\,v^{9}}{1\,451\,862\,491\,496\,617\,710\,284\,533\,236\,465\,918\,746\,258\,432}-\frac{6\,667\,682\,461\,303\,399\,577\,412\,194\,337\,865\,450\,141\,v^{10}}{164\,984\,374\,033\,706\,557\,986\,878\,776\,871\,127\,130\,256\,640\,000}-\frac{218\,341\,245\,241\,156\,182\,731\,852\,048\,841\,475\,277\,537\,v^{11}}{27\,222\,421\,715\,561\,582\,067\,834\,998\,183\,735\,976\,492\,345\,600\,000}+\frac{58\,147\,259\,394\,853\,645\,168\,355\,091\,951\,444\,967\,v^{12}}{24\,747\,656\,105\,055\,983\,698\,031\,816\,530\,669\,069\,538\,496\,000}-\frac{204\,557\,261\,814\,357\,583\,533\,765\,948\,757\,024\,811\,v^{13}}{1\,361\,121\,085\,778\,079\,103\,391\,749\,909\,186\,798\,824\,617\,280\,000}+\frac{20\,448\,146\,081\,015\,322\,202\,191\,566\,828\,053\,987\,v^{14}}{412\,460\,935\,084\,266\,394\,967\,196\,942\,177\,817\,825\,641\,600\,000}+\frac{95\,431\,254\,893\,551\,425\,425\,331\,949\,385\,494\,799\,v^{15}}{68\,056\,054\,288\,903\,955\,169\,587\,495\,459\,339\,941\,230\,864\,000\,000}-\frac{2\,203\,225\,472\,459\,394\,296\,714\,697\,057\,863\,v^{16}}{6\,135\,782\,505\,385\,781\,082\,156\,648\,726\,612\,166\,001\,280\,000}+\frac{14\,300\,489\,696\,519\,671\,276\,503\,594\,466\,459\,v^{17}}{337\,468\,037\,796\,217\,959\,518\,615\,679\,963\,669\,130\,070\,400\,000}-\frac{146\,532\,491\,389\,849\,560\,277\,711\,780\,104\,023\,v^{18}}{12\,373\,828\,052\,527\,991\,849\,015\,908\,265\,334\,534\,769\,248\,000\,000}+\frac{109\,727\,114\,703\,154\,096\,983\,447\,147\,223\,069\,v^{19}}{2\,041\,681\,628\,667\,118\,655\,087\,624\,863\,780\,198\,236\,925\,920\,000\,000},\ \cdots\cdots\right\}
$$

Step 3 . V(x)の解を原始元vで表す

In[50]:=
```
vs /. {α → α', β → β', γ → γ', δ → δ', ϵ → ϵ'} // Simplify; (*vsの要素をvで表す*)
PolynomialMod[V[#], V[v]] & /@ %;
Position[%, 0] // Flatten
vset = vs[[%]] /. {α → α', β → β', γ → γ', δ → δ', ϵ → ϵ'} // Collect[#, v] &;
vset // Short
```

Out[52]=
```
{1, 11, 14, 24, 30, 34, 37, 45, 51, 56, 65, 70, 76, 84, 87, 91, 97, 107, 110, 120}
```

Out[54]//Short=
```
{v, <<18>>, -v}
```

Step4 . Galois群を求める

In[55]:=
```
sols = {α', β', γ', δ', ϵ'};
Table[
  perm = sols /. {vset[[1]] → vset[[i]]} // PolynomialMod[#, V[v]] &;
  Table[FirstPosition[Simplify[perm - sols[[k]] {1, 1, 1, 1, 1}], 0][[1]], {k, 1, 5}] //
   Flatten, {i, 1, Length[vset]}];
MatrixForm[%]
```

Out[57]//MatrixForm=

1	2	3	4	5
1	4	2	5	3
1	3	5	2	4
1	5	4	3	2
2	1	5	4	3
5	1	2	3	4
3	1	4	2	5
4	1	3	5	2
2	4	1	3	5
3	2	1	5	4
4	5	1	2	3
5	3	1	4	2
2	5	3	1	4
5	2	4	1	3
4	3	2	1	5
3	4	5	1	2
2	3	4	5	1
4	2	5	3	1
3	5	2	4	1
5	4	3	2	1

上はガロア流の表現．現代の書き方に直すと次の様になる．
Gal={*Id*, (2,4,5,3), (2,3,5,4), (2,5)(3,4), (1,2)(3,5), (1,5,4,3,2),
(1,3,4,2), (1,4,5,2), (1,2,4,3), (1,3)(4,5), (1,4,2,5,3), (1,5,2,3), (1,2,5,4),
(1,5,3,4), (1,4)(2,3), (1,3,5,2,4), (1,2,3,4,5), (1,4,3,5), (1,3,2,5), (1,5)(2,4)}

ここで$\sigma = (1, 2, 3, 4, 5)$, $\tau = (2, 3, 4, 5)$とおくと，$\sigma^5 = \tau^4 = Id$ だから
Gal={Id, τ^3, τ, τ^2, $\tau^2\sigma^4$, σ^4, $\tau^3\sigma^4$, $\tau\sigma^4$, $\tau^2\sigma^3$, σ^3, $\tau^3\sigma^3$, $\tau^3\sigma^2$, $\tau\sigma^2$, $\tau^2\sigma^2$, σ^2, σ, $\tau^3\sigma$, $\tau\sigma$, $\tau^2\sigma$}=<τ,σ> $\simeq$ $C_4\, C_5$

このように $C_4\, C_5$ の形で書ける群を F_{20} (フロベニウス群) というらしい.
分解列は　$F_{20} \triangleright D_5 \triangleright C_5 \triangleright \{Id\}$
ただし C_n は巡回群, D_n は2面体群で
$C_5 = <\sigma>$, $D_5 = C_5 + \tau^2 C_5$
$F_{20} = D_5 + \tau D_5 = C_5 + \tau C_5 + \tau^2 C_5 + \tau^3 C_5 \simeq C_4\, C_5$

[part2.F20_Solution.nb （厳密解を求める) に続く]