x^5 -5 x + 12 のGalois 群 with *Mathematica* 13.3

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「FindGaloisGroup.nb」で最後から2番目に書いた「StepByStep Mathematica Program 」を 5次向けに直し，それを使って求めます．もちろん 最後に挙げた「Automated Mathematica Program」の方を使うとすぐ求まりますが，それでは説明になりません．いまさら「説明は不要」という方は C5_Solutions.nb/pdf を御覧ください． 逆に，より詳しい説明は「FindGaloisGroup.nb/pdf」をご覧ください．

Step1 . 分解方程式

(出力は 120次の方程式R (x)を因数分解した式, そして分解方程式V(x)です． 分解方程式はR(x)の他の因子を選んでも大丈夫です)

```
In[1]:= ClearAll["`*"];
f[x_] = x^5 - 5 x + 12;
vars = {α, β, γ, δ, ϵ};
vs = Permutations[vars].{5, 2, 3, 4, 1}; (*R(x)に重解があるときはここを変える*)
coef = CoefficientList[f[x], x] // Reverse // Drop[#, 1] &;
{s1, s2, s3, s4, s5} = Table[SymmetricPolynomial[i, vars], {i, 1, 5}];
contents =
  {s1 + coef〚1〛, s2 - coef〚2〛, s3 + coef〚3〛, s4 - coef〚4〛, s5 + coef〚5〛, v - vs〚1〛};
GroebnerBasis[contents, {α, β, γ, δ, ϵ, v}]〚1〛;
Factor[%]
V[x_] = (List @@ %)〚1〛 /. {v → x}
```

Out[9]=

$$\left(148996000-1521500 v^2-12500 v^4+3325 v^6-110 v^8+v^{10}\right)\left(74554000+6730000 v+1323500 v^2-1435000 v^3-12500 v^4+26000 v^5+3325 v^6-200 v^7-90 v^8+v^{10}\right)\left(74554000-6730000 v+1323500 v^2+1435000 v^3-12500 v^4-26000 v^5+3325 v^6+200 v^7-90 v^8+v^{10}\right)\left(1600000+5462500 v^2-87500 v^4+5125 v^6-50 v^8+v^{10}\right)\left(3546000-1290000 v+2123500 v^2-1125000 v^3+147500 v^4+6000 v^5+1325 v^6-600 v^7-10 v^8+v^{10}\right)\left(3546000+1290000 v+2123500 v^2+1125000 v^3+147500 v^4-6000 v^5+1325 v^6+600 v^7-10 v^8+v^{10}\right)\left(4954000-6510000 v+3743500 v^2-975000 v^3+107500 v^4-6000 v^5+3325 v^6-600 v^7+10 v^8+v^{10}\right)\left(4954000+6510000 v+3743500 v^2+975000 v^3+107500 v^4+6000 v^5+3325 v^6+600 v^7+10 v^8+v^{10}\right)\left(900000-537500 v^2+87500 v^4+125 v^6+50 v^8+v^{10}\right)\left(1546000-1730000 v+963500 v^2-265000 v^3+77500 v^4-10000 v^5+3325 v^6-200 v^7+90 v^8+v^{10}\right)\left(1546000+1730000 v+963500 v^2+265000 v^3+77500 v^4+10000 v^5+3325 v^6+200 v^7+90 v^8+v^{10}\right)\left(2304000+518500 v^2+72500 v^4+4325 v^6+110 v^8+v^{10}\right)$$

Out[10]=

$$148996000-1521500 x^2-12500 x^4+3325 x^6-110 x^8+x^{10}$$

Step2 . f (x)の解を原始元vで表す

In[11]:=

```
GroebnerBasis[contents, {α, v}, {β, γ, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
α' = α /. Solve[% == 0, α][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {β, v}, {α, γ, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
β' = β /. Solve[% == 0, β][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {γ, v}, {α, β, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
γ' = γ /. Solve[% == 0, γ][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {δ, v}, {α, β, γ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
δ' = δ /. Solve[% == 0, δ][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {ϵ, v}, {α, β, γ, δ}] // PolynomialMod[Last[#], V[v]] &;
ϵ' = ϵ /. Solve[% == 0, ϵ][[1]] // Collect[#, v] &;
{α', β', γ', δ', ϵ'}
```

Out[21]=

$$\left\{\frac{33144631}{59842152}+\frac{2652763177\,v}{11549535336}-\frac{1597015\,v^2}{159579072}-\frac{54622489\,v^3}{30798760896}-\frac{6619\,v^4}{9651960}+\frac{339857\,v^5}{7451313120}+\frac{63233\,v^6}{3191581440}-\frac{66109\,v^7}{615975217920}-\frac{8251\,v^8}{47873721600}-\frac{15653\,v^9}{1847925653760},\right.$$

$$-\frac{6522187}{29921076}-\frac{234620657\,v}{5774767668}-\frac{670241\,v^2}{79789536}-\frac{54622489\,v^3}{15399380448}+\frac{20767\,v^4}{19303920}+\frac{339857\,v^5}{3725656560}-\frac{21773\,v^6}{1595790720}-\frac{66109\,v^7}{307987608960}-\frac{37\,v^8}{4787372160}-\frac{15653\,v^9}{923962826880},$$

$$-\frac{20100257}{29921076}+\frac{2937497\,v^2}{79789536}-\frac{7529\,v^4}{9651960}-\frac{19687\,v^6}{1595790720}+\frac{8621\,v^8}{23936860800},$$

$$-\frac{6522187}{29921076}+\frac{234620657\,v}{5774767668}-\frac{670241\,v^2}{79789536}+\frac{54622489\,v^3}{15399380448}+\frac{20767\,v^4}{19303920}-\frac{339857\,v^5}{3725656560}-\frac{21773\,v^6}{1595790720}+\frac{66109\,v^7}{307987608960}-\frac{37\,v^8}{4787372160}+\frac{15653\,v^9}{923962826880},$$

$$\left.\frac{33144631}{59842152}-\frac{2652763177\,v}{11549535336}-\frac{1597015\,v^2}{159579072}+\frac{54622489\,v^3}{30798760896}-\frac{6619\,v^4}{9651960}-\frac{339857\,v^5}{7451313120}+\frac{63233\,v^6}{3191581440}+\frac{66109\,v^7}{615975217920}-\frac{8251\,v^8}{47873721600}+\frac{15653\,v^9}{1847925653760}\right\}$$

Step 3 . V(x)の解を原始元vで表す

In[22]:=
```
vs /. {α → α', β → β', γ → γ', δ → δ', ϵ → ϵ'} // Simplify; (*vsの要素をvで表す*)
PolynomialMod[V[#], V[v]] & /@ %;
Position[%, 0] // Flatten
vset = vs⟦%⟧ /. {α → α', β → β', γ → γ', δ → δ', ϵ → ϵ'} // Collect[#, v] &
```

Out[24]=

{1, 24, 30, 34, 56, 65, 87, 91, 97, 120}

Out[25]=

$$\Big\{v,\ \frac{19032865}{59842152}+\frac{8896772159\,v}{11549535336}+\frac{6958943\,v^2}{159579072}+\frac{54622489\,v^3}{30798760896}-\frac{6983\,v^4}{1930392}-\frac{339857\,v^5}{7451313120}+\frac{110951\,v^6}{3191581440}+\frac{66109\,v^7}{615975217920}+\frac{9731\,v^8}{47873721600}+\frac{15653\,v^9}{1847925653760},$$

$$\frac{8123275}{59842152}-\frac{3122004491\,v}{11549535336}-\frac{14174419\,v^2}{159579072}-\frac{54622489\,v^3}{30798760896}+\frac{21131\,v^4}{3860784}+\frac{339857\,v^5}{7451313120}-\frac{115123\,v^6}{3191581440}-\frac{66109\,v^7}{615975217920}-\frac{27343\,v^8}{47873721600}-\frac{15653\,v^9}{1847925653760},$$

$$-\frac{19032865}{59842152}-\frac{8896772159\,v}{11549535336}-\frac{6958943\,v^2}{159579072}-\frac{54622489\,v^3}{30798760896}+\frac{6983\,v^4}{1930392}+\frac{339857\,v^5}{7451313120}-\frac{110951\,v^6}{3191581440}-\frac{66109\,v^7}{615975217920}-\frac{9731\,v^8}{47873721600}-\frac{15653\,v^9}{1847925653760},$$

$$-\frac{8123275}{59842152}-\frac{3122004491\,v}{11549535336}+\frac{14174419\,v^2}{159579072}-\frac{54622489\,v^3}{30798760896}-\frac{21131\,v^4}{3860784}+\frac{339857\,v^5}{7451313120}+\frac{115123\,v^6}{3191581440}-\frac{66109\,v^7}{615975217920}+\frac{27343\,v^8}{47873721600}-\frac{15653\,v^9}{1847925653760},$$

$$-\frac{8123275}{59842152}+\frac{3122004491\,v}{11549535336}+\frac{14174419\,v^2}{159579072}+\frac{54622489\,v^3}{30798760896}-\frac{21131\,v^4}{3860784}-\frac{339857\,v^5}{7451313120}+\frac{115123\,v^6}{3191581440}+\frac{66109\,v^7}{615975217920}+\frac{27343\,v^8}{47873721600}+\frac{15653\,v^9}{1847925653760},$$

$$-\frac{19032865}{59842152}+\frac{8896772159\,v}{11549535336}-\frac{6958943\,v^2}{159579072}+\frac{54622489\,v^3}{30798760896}+\frac{6983\,v^4}{1930392}-\frac{339857\,v^5}{7451313120}-\frac{110951\,v^6}{3191581440}+\frac{66109\,v^7}{615975217920}-\frac{9731\,v^8}{47873721600}+\frac{15653\,v^9}{1847925653760},$$

$$\frac{8123275}{59842152}+\frac{3122004491\,v}{11549535336}-\frac{14174419\,v^2}{159579072}+\frac{54622489\,v^3}{30798760896}+\frac{21131\,v^4}{3860784}-\frac{339857\,v^5}{7451313120}-\frac{115123\,v^6}{3191581440}+\frac{66109\,v^7}{615975217920}-\frac{27343\,v^8}{47873721600}+\frac{15653\,v^9}{1847925653760},$$

$$\frac{19032865}{59842152}-\frac{8896772159\,v}{11549535336}+\frac{6958943\,v^2}{159579072}-\frac{54622489\,v^3}{30798760896}-\frac{6983\,v^4}{1930392}+\frac{339857\,v^5}{7451313120}+\frac{110951\,v^6}{3191581440}-\frac{66109\,v^7}{615975217920}+\frac{9731\,v^8}{47873721600}-\frac{15653\,v^9}{1847925653760},\ -v\Big\}$$

Step4 . Galois群を求める

In[26]:=
```
sols = {α', β', γ', δ', ϵ'};
Table[
  perm = sols /. {vset⟦1⟧ → vset⟦i⟧} // PolynomialMod[#, V[v]] &;
  Table[FirstPosition[Simplify[perm - sols⟦k⟧ {1, 1, 1, 1, 1}], 0]⟦1⟧, {k, 1, 5}] //
   Flatten, {i, 1, Length[vset]}];
MatrixForm[%]
```

Out[28]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \\ 2 & 1 & 5 & 4 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 5 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

現代の書き方では次の様になる.

{*Id*, (2, 5) (3, 4), (1, 2) (3, 5),
 (1, 5, 4, 3, 2), (1, 3) (4, 5), (1, 4, 2, 5, 3),
 (1, 4) (2, 3), (1, 3, 5, 2, 4), (1, 2, 3, 4, 5), (1, 5) (2, 4)}

ここで σ = (1, 2, 3, 4, 5), τ = (2, 3, 5, 4) とすると,

上の群は τ^2 = (2, 3) (4, 5) とσで生成される2面体群 D_5 と一致する.

C_5 を5次巡回群とすると, $D_5 = C_5 + \tau^2 C_5$

分解列は $D_5 \triangleright C_5 \triangleright \{Id\}$

[part2 . **D5_Solution.nb** (厳密解を求める) に続く]