$x^5 - 10\,x^3 + 5\,x^2 + 10x + 1$ の Galois群 with *Mathematica* 13.3

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「FindGaloisGroup.nb」で最後から2番目に書いた「StepByStep Mathematica Program 」を 5次向けに直し，それを使って求めます．もちろん 最後に挙げた「Automated Mathematica Program」の方を使うとすぐ求まりますが，それでは説明になりません．いまさら「説明は不要」という方は C5_Solutions.nb/pdf を御覧ください． 逆に，より詳しい説明は「FindGaloisGroup.nb/pdf」をご覧ください．

Step1 . 分解方程式

(出力は 120次の方程式R (x)を因数分解した式, そして分解方程式V(x)です． 分解方程式はR(x)の他の因子を選んでも大丈夫です)

In[ ]:=
```
ClearAll["`*"];
f[x_] = x^5 - 10 x^3 + 5 x^2 + 10 x + 1;
vars = {α, β, γ, δ, ϵ};
vs = Permutations[vars].{1, 2, 3, 4, 5}; (*R(x)に重解があるときはここを変える*)
coef = CoefficientList[f[x], x] // Reverse // Drop[#, 1] &;
{s1, s2, s3, s4, s5} = Table[SymmetricPolynomial[i, vars], {i, 1, 5}];
contents =
   {s1 + coef⟦1⟧, s2 - coef⟦2⟧, s3 + coef⟦3⟧, s4 - coef⟦4⟧, s5 + coef⟦5⟧, v - vs⟦1⟧};
GroebnerBasis[contents, {α, β, γ, δ, ϵ, v}]⟦1⟧;
Factor[%]
V[x_] = (List @@ %)⟦1⟧ /. {v → x}
```

Out[ ]=

$$\left(-4375+2500 v-125 v^3+v^5\right)\left(4375+2500 v-125 v^3+v^5\right)\left(-1225-1750 v-800 v^2-125 v^3+v^5\right)\left(625-625 v-625 v^2-125 v^3+v^5\right)\left(175-125 v-525 v^2-125 v^3+v^5\right)\left(2525+1125 v-425 v^2-125 v^3+v^5\right)\left(-175+750 v-350 v^2-125 v^3+v^5\right)\left(-175+2125 v-225 v^2-125 v^3+v^5\right)\left(7525+3375 v-175 v^2-125 v^3+v^5\right)\left(8575+3500 v-100 v^2-125 v^3+v^5\right)\left(4825+3750 v-100 v^2-125 v^3+v^5\right)\left(4975+2875 v-75 v^2-125 v^3+v^5\right)\left(-1225+1250 v-50 v^2-125 v^3+v^5\right)\left(1225+1250 v+50 v^2-125 v^3+v^5\right)\left(-4975+2875 v+75 v^2-125 v^3+v^5\right)\left(-8575+3500 v+100 v^2-125 v^3+v^5\right)\left(-4825+3750 v+100 v^2-125 v^3+v^5\right)\left(-7525+3375 v+175 v^2-125 v^3+v^5\right)\left(175+2125 v+225 v^2-125 v^3+v^5\right)\left(175+750 v+350 v^2-125 v^3+v^5\right)\left(-2525+1125 v+425 v^2-125 v^3+v^5\right)\left(-175-125 v+525 v^2-125 v^3+v^5\right)\left(-625-625 v+625 v^2-125 v^3+v^5\right)\left(1225-1750 v+800 v^2-125 v^3+v^5\right)$$

Out[ ]=

$$-4375+2500 x-125 x^3+x^5$$

Step2 . f (x)の解を原始元vで表す

In[∘]:=

```
GroebnerBasis[contents, {α, v}, {β, γ, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
α' = α /. Solve[% == 0, α][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {β, v}, {α, γ, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
β' = β /. Solve[% == 0, β][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {γ, v}, {α, β, δ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
γ' = γ /. Solve[% == 0, γ][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {δ, v}, {α, β, γ, ϵ}] // PolynomialMod[Last[#], V[v]] &;
δ' = δ /. Solve[% == 0, δ][[1]] // Collect[#, v] &;
GroebnerBasis[contents, {ϵ, v}, {α, β, γ, δ}] // PolynomialMod[Last[#], V[v]] &;
ϵ' = ϵ /. Solve[% == 0, ϵ][[1]] // Collect[#, v] &;
{α', β', γ', δ', ϵ'}
```

Out[∘]=

$$\left\{\frac{282}{43}-\frac{41\,v}{43}-\frac{103\,v^2}{215}+\frac{v^3}{215}+\frac{22\,v^4}{5375},\ -\frac{788}{43}+\frac{135\,v}{43}+\frac{293\,v^2}{215}-\frac{28\,v^3}{1075}-\frac{63\,v^4}{5375},\ \frac{332}{43}-\frac{87\,v}{43}-\frac{142\,v^2}{215}+\frac{19\,v^3}{1075}+\frac{32\,v^4}{5375},\ \frac{572}{43}-\frac{110\,v}{43}-\frac{183\,v^2}{215}+\frac{26\,v^3}{1075}+\frac{37\,v^4}{5375},\ -\frac{398}{43}+\frac{103\,v}{43}+\frac{27\,v^2}{43}-\frac{22\,v^3}{1075}-\frac{28\,v^4}{5375}\right\}$$

Step 3 . V(x)の解を原始元vで表す

In[∘]:=

```
vs /. {α → α', β → β', γ → γ', δ → δ', ϵ → ϵ'} // Simplify; (*vsの要素をvで表す*)
PolynomialMod[V[#], V[v]] & /@ %;
Position[%, 0] // Flatten
vset = vs[[%]] /. {α → α', β → β', γ → γ', δ → δ', ϵ → ϵ'} // Collect[#, v] &
```

Out[∘]=

{1, 34, 65, 91, 97}

Out[∘]=

$$\left\{v,\ \frac{1410}{43}-\frac{162\,v}{43}-\frac{103\,v^2}{43}+\frac{v^3}{43}+\frac{22\,v^4}{1075},\ -\frac{2530}{43}+\frac{513\,v}{43}+\frac{190\,v^2}{43}-\frac{23\,v^3}{215}-\frac{41\,v^4}{1075},\ -\frac{870}{43}+\frac{78\,v}{43}+\frac{48\,v^2}{43}-\frac{4\,v^3}{215}-\frac{9\,v^4}{1075},\ \frac{1990}{43}-\frac{472\,v}{43}-\frac{135\,v^2}{43}+\frac{22\,v^3}{215}+\frac{28\,v^4}{1075}\right\}$$

Step4 . Galois群を求める

In[ ]:=
```
sols = {α', β', γ', δ', ϵ'};
Table[
  perm = sols /. {vset[[1]] → vset[[i]]} // PolynomialMod[#, V[v]] &;
  Table[FirstPosition[Simplify[perm - sols[[k]] {1, 1, 1, 1, 1}], 0][[1]], {k, 1, 5}] //
   Flatten, {i, 1, 5}];
MatrixForm[%]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

現代の書き方では $\mathrm{Gal} = \{\mathrm{Id}, \sigma, \sigma^2, \sigma^3, \sigma^4\} = <\sigma>, \sigma = (1, 2, 3, 4, 5)$

Galは 5 次巡回群C_5 と一致する．分解列は $C_5 \triangleright \{Id\}$

[part2 . C5_Solution.nb（厳密解を求める）に続く]