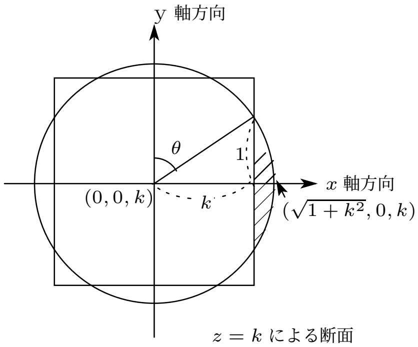
【解答】



平面 $z=k(0 \leqq k \leqq 1)$ による断面積を $S(k)$ とする。$z=k$ 上で、図の様に θ をとると、

$$k=\tan\theta, dk=\frac{1}{\cos^2\theta}d\theta, \begin{array}{c|c} k & 0 \longrightarrow 1 \\ \hline \theta & 0 \longrightarrow \frac{\pi}{4} \end{array}$$

図より

$$S=\frac{(\sqrt{1+k^2})^2}{2}\left(\frac{\pi}{2}-\theta\right)\times 2-k=\left(\frac{\pi}{2}-\theta\right)(1+\tan^2\theta)-\tan\theta$$

$$\therefore\ V=\int_0^{\frac{\pi}{4}}\left\{\frac{\pi(1+\tan^2\theta)}{2}-\tan\theta\right\}\frac{1}{\cos^2\theta}d\theta-\int_0^{\frac{\pi}{4}}\theta(1+\tan^2\theta)\cdot\frac{1}{\cos^2\theta}d\theta$$

右辺の第 1 項、第 2 項をそれぞれ I_1, I_2 とおくと

$$\begin{aligned} I_1&=\int_0^{\frac{\pi}{4}}\left\{\frac{\pi(1+\tan^2\theta)}{2}-\tan\theta\right\}(\tan\theta)'d\theta \\ &=\left[\frac{\pi}{2}\left(\tan\theta+\frac{\tan^3\theta}{3}\right)-\frac{\tan^2\theta}{2}\right]_0^{\frac{\pi}{4}}=\frac{2\pi}{3}-\frac{1}{2} \qquad \cdots① \end{aligned}$$

$$\begin{aligned} I_2&=\int_0^{\frac{\pi}{4}}\theta(1+\tan^2\theta)\cdot(\tan\theta)'d\theta=\int_0^{\frac{\pi}{4}}\theta\left(\tan\theta+\frac{\tan^3\theta}{3}\right)'d\theta \\ &=\left[\theta\left(\tan\theta+\frac{\tan^3\theta}{3}\right)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\left(\tan\theta+\frac{\tan^3\theta}{3}\right)d\theta \end{aligned}$$

ここで

$$\int\tan^3\theta d\theta=\int\tan\theta\cdot\tan^2\theta d\theta=\int\tan\theta\left(-1+\frac{1}{\cos^2\theta}\right)d\theta$$

であるから

$$\begin{aligned} I_2&=\frac{\pi}{4}\left(1+\frac{1}{3}\right)-\int_0^{\frac{\pi}{4}}\frac{2}{3}\tan\theta d\theta-\int_0^{\frac{\pi}{4}}\frac{\tan\theta}{3}(\tan\theta)'d\theta \\ &=\frac{\pi}{3}+\frac{2}{3}\Big[\log|\cos\theta|\Big]_0^{\frac{\pi}{4}}-\left[\frac{\tan^2\theta}{6}\right]_0^{\frac{\pi}{4}}=\frac{\pi}{3}-\frac{1}{6}-\frac{\log 2}{3} \qquad \cdots② \end{aligned}$$

①,② より

$$V=\left(\frac{2\pi}{3}-\frac{1}{2}\right)-\left(\frac{\pi}{3}-\frac{1}{6}-\frac{\log 2}{3}\right)=\frac{1}{3}(\pi-1+\log 2) \qquad \cdots(答)$$