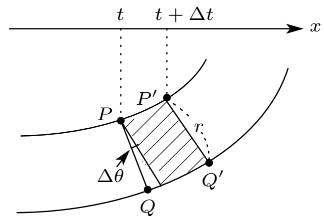
【解答】



(1) $\alpha \leqq x \leqq t(\leqq \beta)$ における線分 PQ の通過領域の面積を $S(t)$,t が Δt だけ増加したときの $S(t)$ と θ の増加量をそれぞれ $\Delta S, \Delta\theta$ とおくと、図より

$$\Delta S = \frac{r^2}{2}\Delta\theta + r\sqrt{1+(f'(t))^2}\Delta t$$

$$\therefore\ S = \int_\alpha^\beta r\sqrt{1+(f'(x))^2}dx + \frac{r^2}{2}(\theta(\beta)-\theta(\alpha)) \qquad \text{Q.E.D}$$

(2) $f(x) = \frac{e^x+e^{-x}}{2}$ のとき

$$f'(x) = \frac{e^x-e^{-x}}{2}$$

したがって $f'(0)=0,\ f'(\log(1+\sqrt{2}))=1$ となるから

$$\theta(\log(1+\sqrt{2})) = \frac{\pi}{4},\ \theta(0)=0$$

また

$$\sqrt{1+(f'(x))^2} = \frac{e^x+e^{-x}}{2}$$

ゆえに

$$S = \int_0^{\log(1+\sqrt{2})} \frac{e^x+e^{-x}}{2}dx + \frac{\pi}{4} - 0 = \left[\frac{e^x-e^{-x}}{2}\right]_0^{\log(1+\sqrt{2})} + \frac{\pi}{4} = 1 + \frac{\pi}{4} \qquad \cdots(\text{答})$$

Comment

(i)r が定数でないとき

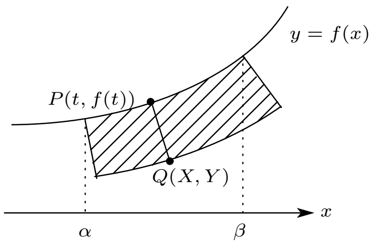
$$S = \int_\alpha^\beta r(x)\sqrt{1+(f'(x))^2}dx + \frac{1}{2}\int_\alpha^\beta \frac{(r(x))^2 f''(x)}{1+(f'(t))^2}dx$$

(ii) 点 Q を $f=f(x)$ の上側にとるとき

$$S = \int_\alpha^\beta r\sqrt{1+(f'(x))^2}dx - \frac{r^2}{2}(\theta(\beta)-\theta(\alpha))$$

と推定されます。

【別解】(1)



$P(t,f(t)),Q(X,Y)$ とおくと

$$\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}t\\f(t)\end{pmatrix}+\frac{r}{\sqrt{1+(f'(t))^2}}\begin{pmatrix}f'(t)\\-1\end{pmatrix}$$

$$\begin{aligned}S=\int_\alpha^\beta f(t)dt&+\left\{f(\beta)+f(\beta)-\frac{r}{\sqrt{1+(f'(\beta))^2}}\right\}\frac{rf'(\beta)}{2\sqrt{1+(f'(\beta))^2}}\\&-\left\{f(\alpha)+f(\alpha)-\frac{r}{\sqrt{1+(f'(\alpha))^2}}\right\}\frac{rf'(\alpha)}{2\sqrt{1+(f'(\alpha))^2}}\\&-\int_\alpha^\beta\left\{f(t)-\frac{r}{\sqrt{1+(f'(t))^2}}\right\}\left\{1+r\left(\frac{f'(t)}{\sqrt{1+(f'(t))^2}}\right)'\right\}dt\quad\cdots①\end{aligned}$$

$\left(\frac{f'(t)}{\sqrt{1+(f'(t))^2}}\right)'=\frac{f''(t)}{(\sqrt{1+(f'(t))^2})^3}$ だから第 4 項の積分を I とおくと

$$\begin{aligned}I&=\int_\alpha^\beta f(t)dt+\int_\alpha^\beta rf(t)\left(\frac{f'(t)}{\sqrt{1+(f'(t))^2}}\right)'dt-\int_\alpha^\beta\frac{r}{\sqrt{1+(f'(t))^2}}dt-r^2\int_\alpha^\beta\frac{f''(t)}{(1+(f'(t))^2)^2}dt\\&=\int_\alpha^\beta f(t)dt+\Big[\frac{rf(t)f'(t)}{\sqrt{1+(f'(t))^2}}\Big]_\alpha^\beta-\int_\alpha^\beta\frac{r(f'(t))^2}{\sqrt{1+(f'(t))^2}}dt-\int_\alpha^\beta\frac{r}{\sqrt{1+(f'(t))^2}}dt-r^2\int_\alpha^\beta\frac{f''(t)}{(1+(f'(t)))^2}dt\\&=\int_\alpha^\beta f(t)dt+\frac{rf(\beta)f'(\beta)}{\sqrt{1+(f'(\beta))^2}}-\frac{rf(\alpha)f'(\alpha)}{\sqrt{1+(f'(\alpha))^2}}-r\int_\alpha^\beta\sqrt{1+(f'(t))^2}dt-r^2\int_\alpha^\beta\frac{f''(t)}{(1+(f'(t)))^2}dt\end{aligned}$$

① へ代入して

$$S=r\int_\alpha^\beta\sqrt{1+(f'(t))^2}dt+r^2\int_\alpha^\beta\frac{f''(t)}{(1+(f'(t))^2)^2}dt-\frac{r^2f'(\beta)}{2(1+(f'(\beta))^2)}+\frac{r^2f'(\alpha)}{2(1+(f'(\alpha))^2)}$$

ここで

$$\begin{aligned}&\frac{f''(t)}{(1+(f'(t))^2)^2}-\frac{1}{2}\left(\frac{f'(t)}{1+(f'(t))^2}\right)'\\&\qquad=\frac{f''(t)}{(1+(f'(t))^2)^2}-\frac{f''(t)-f''(t)(f'(t))^2}{2(1+(f'(t))^2)^2}=\frac{f''(t)}{2(1+(f'(t))^2)}=\frac{1}{2}\left(\tan^{-1}f'(t)\right)'\end{aligned}$$

であるから

$$\frac{f''(t)}{(1+(f'(t))^2)^2}=\left(\frac{1}{2}\tan^{-1}f'(t)+\frac{f'(t)}{2(1+(f'(t))^2)}\right)'$$

よって

$$\begin{aligned}S&=r\int_\alpha^\beta\sqrt{1+(f'(t))^2}dt+\Big[\frac{r^2}{2}\tan^{-1}f'(t)+\frac{r^2f'(t)}{2(1+(f'(t))^2)}\Big]_\alpha^\beta-\frac{r^2f'(\beta)}{2(1+(f'(\beta))^2)}+\frac{r^2f'(\alpha)}{2(1+(f'(\alpha))^2)}\\&=r\int_\alpha^\beta\sqrt{1+(f'(t))^2}dt+\frac{r^2}{2}\left\{\tan^{-1}(f'(\beta))-\tan^{-1}(f'(\alpha))\right\}\\&=r\int_\alpha^\beta\sqrt{1+(f'(t))^2}dt+\frac{r^2}{2}\{\theta(\beta)-\theta(\alpha)\}\end{aligned}$$

Q.E.D.