Finding Taylor coefficients *by your eyes* with GeoGebra & MuPAD

ATCM in Kuala Lumpur, Dec18th / 2010

Ogose shigeki (生越　茂樹)
http://mixedmoss.com/atcm/2010/

1. Taylor series

When n-th derivative of $f(x)$ exists in an interval, for x and a in the interval,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + R_n$$

Remainder: $R_n = \dfrac{f^n(\xi)}{n!}(x-a)^n$, $\quad$ *ξ is between x & a*

Examples

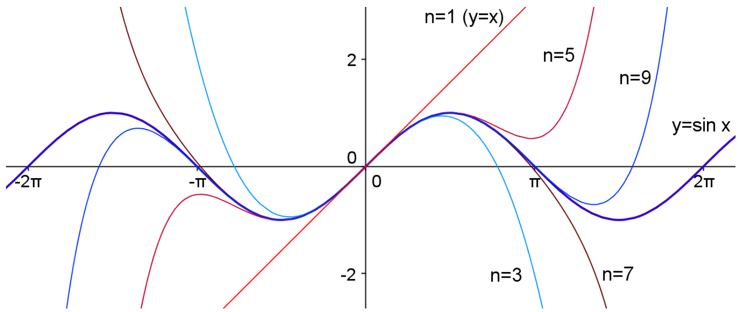
$$\begin{cases} e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^{n-1}}{(n-1)!} + R_n \\ \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - + \cdots + (-1)^{2n-1}\dfrac{x^{2n-1}}{(2n-1)!} + R_n \\ \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - + \cdots + (-1)^{2(n-1)}\dfrac{x^{2(n-1)}}{(2n-2)!} + R_n \\ \log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - + \cdots + (-1)^n\dfrac{x^{n-1}}{n-1} + R_n \end{cases}$$

Ex1. Taylor series of $\sin x$ (around $x=0$)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + (-1)^{2n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots (-\infty < x < \infty)$$

($f_n(x)$ is the first n-th parabola of the series, such as

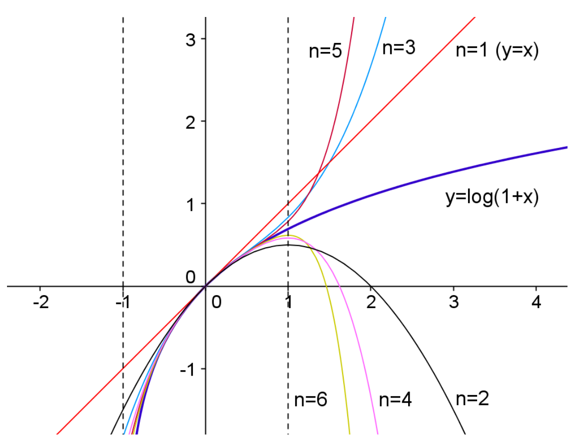
$f_1(x) = x,\ f_3(x) = x - \frac{x^3}{3!},\ f_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!},\ \cdots$)



sinx.ggb

Ex2. Taylor series of $\log(1+x)$ (around $x=0$)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \ (-1 < x < 1)$$



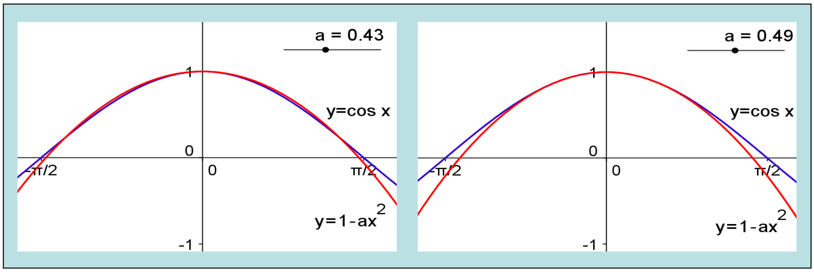
log(1+x).ggb

2. Finding Taylor coefficients *by your eyes*.

Ex1. Taylor series of $\cos x$ (order 2 around $x=0$)

Tangent to $y=f(x)=\cos x$ at (0,1) is "$y=1$". Therefore $\boxed{\cos x \approx 1 \text{ (1st approx)}}$

To find the 2nd approx, compare $R_2(x) \equiv f(x) - \text{(1st approximation)} = \cos x - 1$ & ax^2.



"Near $x=0$", the right one ($a=0.49$) approximates $f(x)$ better than the left.

Therefore you can tell " when $x \approx 0$, $\cos x - 1 \approx 0.49x^2 \approx \frac{1}{2}x^2$."

That is $\boxed{\cos x \approx 1 + \frac{1}{2}x^2 \text{ (2nd approx)}}$

taylor.mn

[Rough Proof]

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1+\cos x}$$

$$= \frac{1}{2}$$

Therefore , near x=0,

$$\frac{1-\cos x}{x^2} \approx \frac{1}{2} \iff 1-\cos x \approx \frac{1}{2}x^2 \iff \cos x \approx 1 - \frac{1}{2}x^2$$

That is,

$$\boxed{\cos x \approx 1 - \frac{1}{2}x^2 \quad (x \approx 0)} \text{ (2nd order Taylor series)}$$

Ex2. Taylor series of $f(x)=e^x$ (order 3 around $x=0$)

taylor.mn

I. Since tangent to $y=e^x$ at (0,1) is $y=1+x$, $\boxed{e^x \approx 1+x \text{ (1st approx)}}$

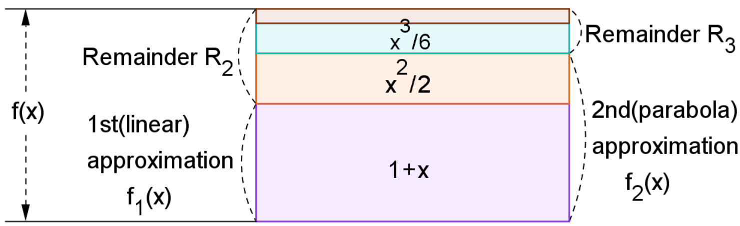
II. To get the 2nd approx, compare $R_2(x) \equiv f(x)-(\text{1st approx}) = e^x-(1+x)$ & ax^2.

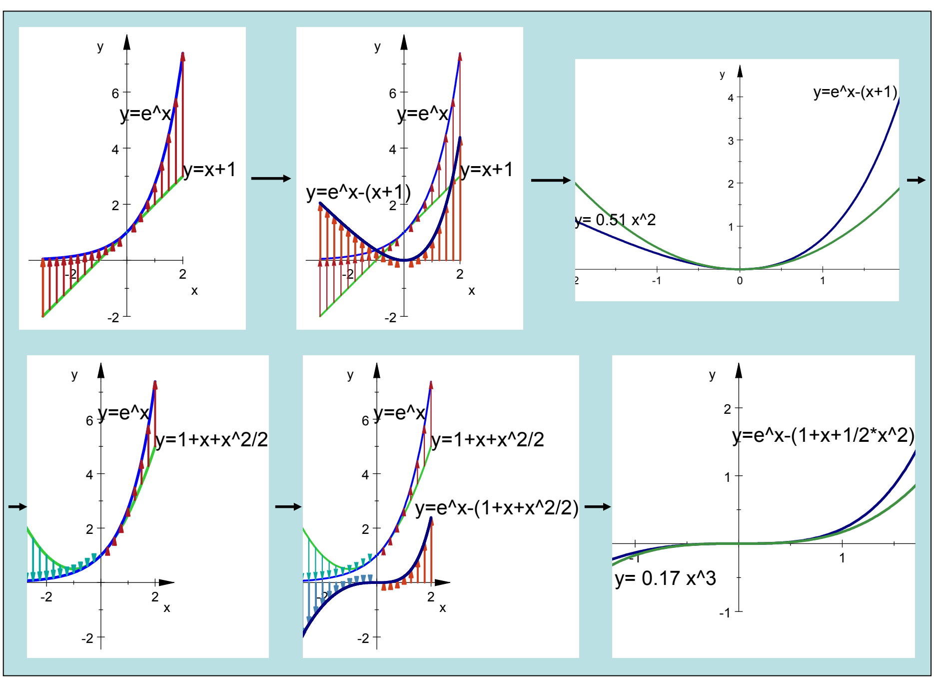
Near $x=0$, $y=\frac{1}{2}x^2$ *looks* closest to $R_2(x)$, thus $\boxed{e^x \approx 1+x+\frac{1}{2}x^2 \text{ (2nd approx)}}$

III. To get the 3rd approx, compare $R_3(x) \equiv f(x)-(\text{2nd approx}) = e^x-\left(1+x+\frac{1}{2}x^2\right)$ & ax^3.

Near $x=0$, $y=\frac{1}{6}x^3$ *looks* closest to $R_3(x)$, thus $\boxed{e^x \approx 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3 \text{ (3rd approx)}}$

⋮ ⋮ ⋮





3. A proof of Taylor series

3-1. Caushy's mean-value theorem

If $f(x)$, $g(x)$ are differentiable on (a,b) and continuous on $[a,b]$,
Then there exists ξ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)} \quad (a \leqq \xi \leqq b)$$

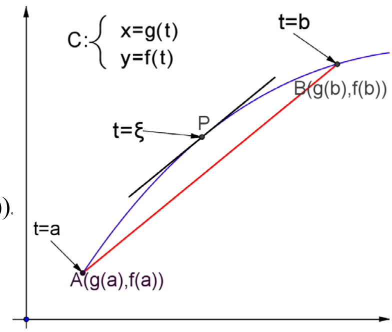
* On condition that, however, $(f'(x))^2 + (g'(x))^2 \neq 0 \ \wedge\ g(a) \neq g(b)$

Geometrical interpretation

Considering a parametric curve:

C: $x = g(t),\ y = f(t)\ (a \leq t \leq b)$

and points $A(g(a), f(a)),\ B(g(b), f(b))$ on C.
Then, the left side is the gradient of line AB,
while the right side is of tangent to C at P$(g(\xi), f(\xi))$.
Thus it means, " there exists a point P on C where tangent line to C is parallel to line AB."



3-2. A proof of Taylor's theorem (n=2&3)

$n = 2$:

Suppose $f''(x)$ exists in an interval around a, then for x in the interval, let

$$R(x) = f(x) - \{f(a) + f'(a)(x-a)\},\quad G(x) = (x-a)^2$$

Since $R(a) = 0,\ R'(a) = 0,$ by Mean-Value theorem

$$\frac{R(x)-R(a)}{G(x)-G(a)} = \frac{f(x) - \{f(a) + f'(a)(x-a)\}}{(x-a)^2}$$

$$= \frac{f'(x_1) - f'(a)}{2(x_1 - a)} \quad (a \leqq x_1 \leqq x)$$

$$= \frac{1}{2} f''(\xi) \quad (a \leqq \xi \leqq x_1)$$

$$\therefore \boxed{f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2!}(x-a)^2 \ (a \leqq \xi \leqq x)}$$

$n = 3:$

Suppose $f^{(3)}(x)$ exists in an interval around a, then for x in the interval, let

$$R(x) = f(x) - \left\{ f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \right\},\ G(x) = (x-a)^3$$

Since $R(a) = 0,\ R'(a) = 0,$ by repetion of Mean-Value theorem

$$\frac{R(x)}{G(x)} = \frac{f(x) - \left\{ f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \right\}}{(x-a)^3}$$

$$= \frac{f'(x_1) - \{ f'(a) + f''(a)(x_1 - a) \}}{3(x_1 - a)^2} \quad (a \leqslant x_1 \leqslant x)$$

$$= \frac{1}{6} \cdot \frac{f''(x_2) - f''(a)}{x_2 - a} \quad (a \leqslant x_2 \leqslant x_1)$$

$$= \frac{1}{6} f'''(\xi) \quad (a \leqslant \xi \leqslant x_2)$$

$$\therefore \boxed{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(\xi)}{3!}(x-a)^3 \ (a \leqslant \xi \leqslant x)}$$